One more sequence that converge to e.

https://www.linkedin.com/feed/update/urn:li:activity:6679626037636198400

$a_n = n\left(\left(1+\frac{1}{n}\right)^n - \left(1+\frac{1}{n+1}\right)^n\right), \forall n \in \mathbb{N}$. Prove that

a) $\frac{n}{n+1}\left(1+\frac{1}{n-1}\right)^{n-1} < a_n < \frac{n}{n+1}\left(1+\frac{1}{n}\right)^{n-1}$;

b) $\lim_{n\to\infty} a_n = e$.

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a) First note that inequality $\frac{n}{n+1}\left(1+\frac{1}{n-1}\right)^{n-1} < a_n$ isn't holds because obvious that $\frac{n}{n+1}\left(1+\frac{1}{n-1}\right)^{n-1} > \frac{n}{n+1}\left(1+\frac{1}{n}\right)^{n-1} \Leftrightarrow \frac{1}{n-1} > \frac{1}{n}$ and inequality in **a**) should be corrected for example as follows:

$$\frac{n}{n+1}\left(1+\frac{1}{n+1}\right)^{n-1} < a_n < \frac{n}{n+1}\left(1+\frac{1}{n}\right)^{n-1}.$$

a1. $a_n < \frac{n}{n+1}\left(1+\frac{1}{n}\right)^{n-1} \Leftrightarrow n\left(\left(1+\frac{1}{n}\right)^n - \left(1+\frac{1}{n+1}\right)^n\right) < \frac{n}{n+1}\left(1+\frac{1}{n}\right)^{n-1} \Leftrightarrow$

$\left(1+\frac{1}{n}\right)^n - \left(1+\frac{1}{n+1}\right)^n < \frac{1}{n+1}\left(1+\frac{1}{n}\right)^{n-1} \Leftrightarrow$

$\left(1+\frac{1}{n}-\frac{1}{n+1}\right)\left(1+\frac{1}{n}\right)^{n-1} < \left(1+\frac{1}{n+1}\right)^n \Leftrightarrow$

$\left(1+\frac{1}{n(n+1)}\right)\left(1+\frac{1}{n}\right)^{n-1} < \left(1+\frac{1}{n+1}\right)^n$ and by AM-GM Inequality

$$\left(1+\frac{1}{n(n+1)}\right)\left(1+\frac{1}{n}\right)^{n-1} \le \left(\frac{1+\frac{1}{n(n+1)}+(n-1)\left(1+\frac{1}{n}\right)}{n}\right)^n = \left(1+\frac{1}{n+1}\right)^n.$$

a2. $\frac{n}{n+1}\left(1+\frac{1}{n+1}\right)^{n-1} < a_n \Leftrightarrow \frac{1}{n+1}\left(\frac{n+2}{n+1}\right)^{n-1} < \left(\frac{n+1}{n}\right)^n - \left(\frac{n+2}{n+1}\right)^n \Leftrightarrow$

$\frac{1}{n+1}\left(\frac{n+2}{n+1}\right)^{n-1} \cdot \left(\frac{n+1}{n+2}\right)^n < \frac{\left(\frac{n+1}{n}\right)^n}{\left(\frac{n+2}{n+1}\right)^n} - 1 \Leftrightarrow \frac{1}{n+2} < \left(1+\frac{1}{n^2+2n}\right)^n - 1$

and by Bernoulli's Inequality $\left(1+\frac{1}{n^2+2n}\right)^n - 1 > 1 + \frac{n}{n^2+2n} - 1 = \frac{1}{n+2}$.

b) by Squeeze Principle immediately follows from **a**).

But we also consider another direct solution of **b**) without using double inequality in **a**)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\left(1+\frac{1}{n+1}\right)^n\left(\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n+1}\right)^n} - 1\right) = e\lim_{n\to\infty} n\left(\left(1+\frac{1}{n^2+2n}\right)^n - 1\right).$$

Let $b_n := n\ln\left(1+\frac{1}{n^2+2n}\right)$. Since $\lim_{n\to\infty} b_n = 0$ (because $0 < n\ln\left(1+\frac{1}{n^2+2n}\right) < n\cdot\frac{1}{n^2+2n} = \frac{1}{n+2}$) then

$$\lim_{n\to\infty} a_n = e\lim_{n\to\infty} n(e^{b_n}-1) = e\lim_{n\to\infty}\left(nb_n\cdot\frac{e^{b_n}-1}{b_n}\right) = e\lim_{n\to\infty} nb_n.$$

Noting that $\lim_{n\to\infty}(n^2+2n)\ln\left(1+\frac{1}{n^2+2n}\right)=1$ we obtain
$\lim_{n\to\infty} nb_n = \lim_{n\to\infty} n^2\ln\left(1+\frac{1}{n^2+2n}\right) =$
$\lim_{n\to\infty}\frac{n}{n+2}\cdot\lim_{n\to\infty}(n^2+2n)\ln\left(1+\frac{1}{n^2+2n}\right)=1$ and, therefore, $\lim_{n\to\infty} a_n = e$.